SAINT IGNATIUS' COLLEGE RIVERVIEW
YEAR 12

MATHEMATICS

Extension One

APRIL
2008

Time allowed – 2 hours
(plus 5 minutes reading time)

Directions to Candidates

1. Attempt **ALL** questions.
2. There are **SEVEN QUESTIONS** of equal value.
3. All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
4. Board-approved calculators may be used.
5. Each question attempted is to be returned in a **SEPARATE BOOKLET** clearly marked Question 1, Question 2, ...etc.
6. Each answer sheet must show your **NAME** and your **TEACHER'S NAME**.

Question One: (12 marks) Please use a separate writing booklet

a) Solve the following equation. 3

$$|2x+4|=9$$

b) If $\sec\theta = \frac{-5}{4}$ and $\tan\theta > 0$, find the exact value of $\cot\theta + \cos\theta$. 3

c) Solve the following equation for x: 2

$$4^{2x-1} = \frac{1}{8}$$

d) An arc of length 2cm subtends an angle of $\frac{\pi}{3}$ at the centre of a circle with radius x cm. Find x. 2

e) The point A $(-2,1)$ is the midpoint of $(a,4)$ and $(-3,b)$. Find a and b. 2

Question Two: (12 marks) Please use a separate writing booklet

a) AB is a chord of length 80cm and is 9cm from the centre of the circle. Find the diameter of the circle. 2

b) Find the following limit. 2

$$\lim_{x \to 0} \frac{2x}{\sin 5x}$$

c) Find the size of the angle between the lines $2x - 3y + 2 = 0$ and $x + 2y - 5 = 0$. Answer to the nearest degree. 3

d) Find the coordinates of the point P(x, y) which divides the interval joining A(6,-4) and B(-1,5) externally in the ratio 2:3. 2

e) Solve for x and graph the solution on a number line. 3

$$\frac{2x+5}{x+1} \geq 3$$

Question Three: (12 marks) Please use a separate writing booklet

a) Determine the value of 3

i) $\tan^{-1}(\sqrt{3})$

ii) $cos^{-1}(\sin\frac{\pi}{3})$

b) If $y = \log_e(x-2)$

i) find the equation of the inverse function 1

ii) state the domain of this inverse function. 2

c) Draw a neat accurate sketch of the following function. Label clearly any key points. 3

$$y = 4\sin^{-1}\frac{x}{2}$$

d) The area bounded by the curve $y = \frac{1}{\sqrt{9+x^2}}$, the x axis and the ordinates $x = -3$ and $x = \sqrt{3}$ is rotated about the x axis. Find the volume of the solid of revolution formed. 3

Question Four: (12 marks) Please use a separate writing booklet

a) i) By using the sum to n terms of an Arithmetic Series show that the sum of the first positive n odd integers is n^2. **1**

ie: $1+3+5+......+(2n-1)=n^2$

ii) Prove the result of part i) by Mathematical Induction. **3**

b) Use Mathematical Induction to show that $5^n \geq 1+4n$ for all positive integers n. **4**

c) Prove by Mathematical Induction that $2n + n^3$ is a multiple of 3, for all positive integers n. **4**

Question Five: (12 marks) Please use a separate writing booklet

a) i) Write the expansion of $\sin(\alpha + \beta)$ 1

ii) Hence, find the exact value of $\sin 105°$ 2

b) Prove the following identity: $\sin 2B = \dfrac{2\tan B}{1+\tan^2 B}$ 3

c) Given that $0<\alpha<\dfrac{\pi}{4}$, prove that $\tan\left(\dfrac{\pi}{4}+\alpha\right) = \dfrac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$ 3

d) Find all the angles β with $0 \le \beta \le 2\pi$ for which 3
$\sin\beta + \cos\beta = 1$

Question Six: (12 marks) Please use a separate writing booklet

a) Find the cartesian equation of the parabola with parametric equations $x = 6t$ and $y = 2t^2$. 1

b) Prove that the normal to the parabola $x^2 = 8y$ at the point $P(4t, 2t^2)$ has equation $x + ty = 2t^3 + 4t$ 3

c) The point $T(t, t^2 + 1)$ lies on the parabola $y = x^2 + 1$.

i) Find the equation of the tangent at T. 2

ii) The tangent at T and the tangent at the vertex meet at *R*. Find the coordinates of *R*. 1

d) On the parabola $x^2 = y$ the point P has coordinates (t, t^2). The equation of the tangent at P is $y - 2tx + t^2 = 0$.

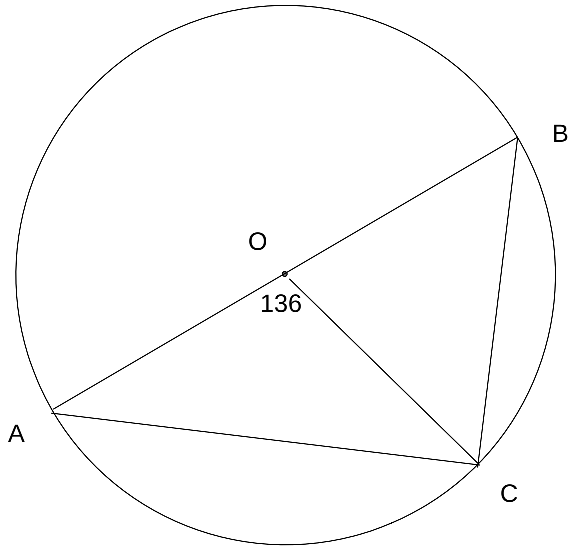
i) State the coordinates of the focus S of the parabola. 1

ii) Prove that the line through S meeting the tangent at right angles in N has equation $4ty + 2x = t$. 2

iii) Hence determine the equation of the locus of N as P varies on the curve. 2

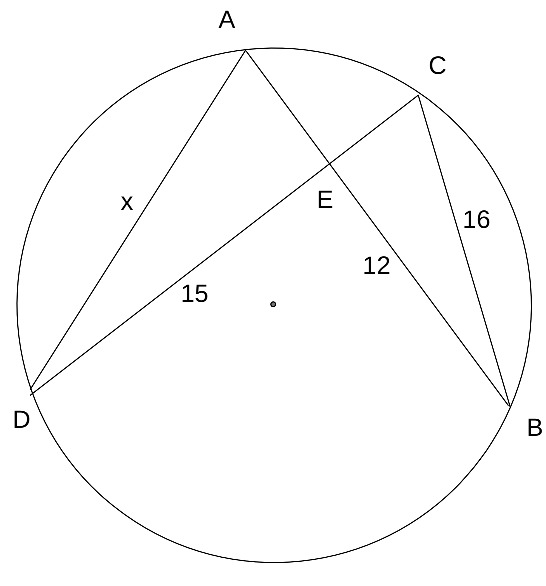
Question Seven: (12 marks) Please use a separate writing booklet

a)



It is known that $\angle AOB$ is straight, $\angle AOC = 136^0$ and O is the centre of the circle. Find the size of $\angle OCB$. 3

b)



i) Prove that ΔAED is similar to ΔCEB 2

ii) Find the value of x 2

c) Two circles touch externally in P. QR is a common tangent to the circles touching the first circle in Q and the second circle in R. The common tangent at P meets QR in S.

i) Draw a clearly labelled diagram of this information. 1

ii) Prove that S is the midpoint of QR 2

iii) Prove that $\angle QPR$ is a right angle 2

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

YR 12 EXT 1 MATHEMATICS

TERM ONE 2008 EXAMINATION.

SOLUTIONS

QUESTION ONE

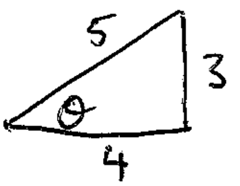
a) $|2x+4| = 9$

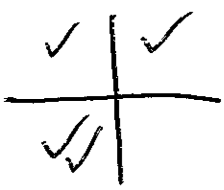
$2x+4=9 \qquad 2x+4=-9$

$2x=5 \qquad 2x=-13$

$x = 5/2 \qquad x = -\frac{13}{2}$

b) $\sec\theta = -\frac{5}{4} \quad \tan\theta > 0$





$\cot\theta + \cos\theta = \frac{4}{3} - \frac{4}{5}$

$= \frac{8}{15}$

c) $4^{2x-1} = \frac{1}{8}$

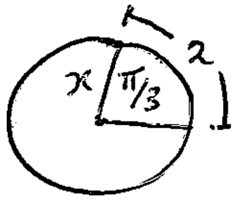
$2^{4x-2} = 2^{3}$

$4x-2 = 3$

$4x = 5$

$x = 5/4$

d)



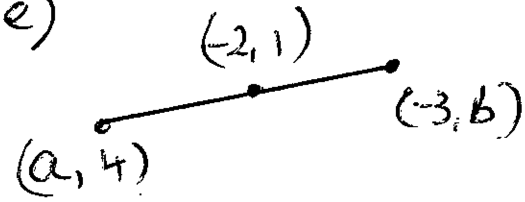
$l = r\theta$

$2 = x \cdot \frac{\pi}{3}$

$\frac{6}{\pi} = x.$

$x = \frac{6}{\pi}$ cm.

e)



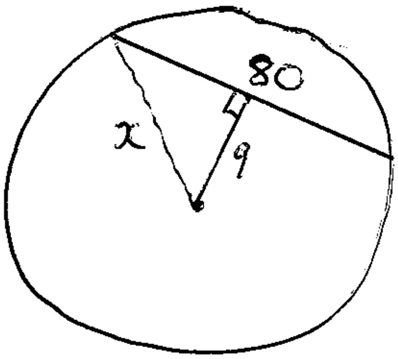
$-2 = \frac{a+(-3)}{2} \qquad 1 = \frac{4+b}{2}$

$-4 = a-3 \qquad 2 = 4+b$

$a = -1 \qquad b = -2$

Question 2

a)



$x^2 = 9^2 + 40^2$

$x = 41$

$\therefore$ DIAMETER $= 82\text{cm}$

b) $\lim\limits_{x \to 0} \dfrac{2x}{\sin 5x}$

$= \dfrac{2}{5} \lim\limits_{x \to 0} \dfrac{5x}{\sin 5x}$

$= \dfrac{2}{5}$

c) $2x - 3y + 2 = 0 \quad m = \dfrac{2}{3}$

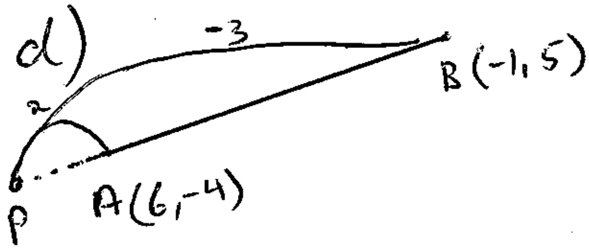
$x + 2y - 5 = 0 \quad m = -\dfrac{1}{2}$

$\tan\theta = \left| \dfrac{\frac{2}{3} + \frac{1}{2}}{1 + \frac{2}{3}(-\frac{1}{2})} \right|$

$= \dfrac{7}{4}$

$\therefore \theta = 60° 15'$

d)



$x = \dfrac{2(-1) - 3(6)}{-1}$

$= 20$

$y = \dfrac{2(5) - 3(-4)}{-1}$

$= -22.$

$P(20, -22)$

e) $\dfrac{2x+5}{x+1} \geq 3 \qquad x \neq -1$

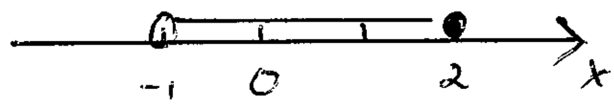
$(2x+5)(x+1) \geq 3(x+1)^2$

$(2x+5)(x+1) - 3(x+1)^2 \geq 0$

$(x+1)(2x+5 - 3(x+1)) \geq 0$

$(x+1)(2x+5-3x-3) \geq 0$

$(x+1)(2-x) \geq 0$



Question 3

a) i) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$

ii) $\cos^{-1}\left(\sin\frac{\pi}{3}\right) = \cos^{-1}\frac{\sqrt{3}}{2}$

$= \frac{\pi}{6}$

b) $y = \log_e(x-2)$

i) $x = \log_e(y-2)$

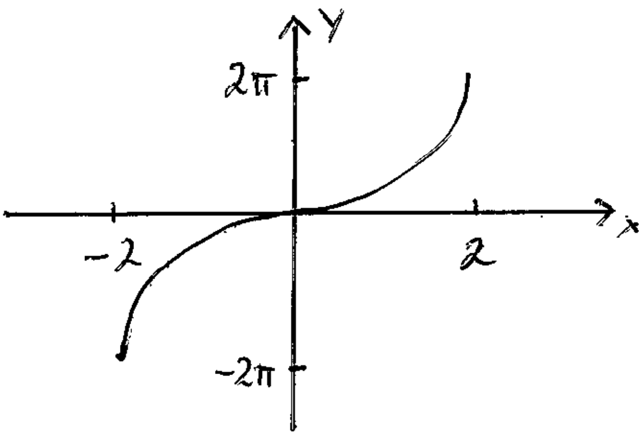
$e^x = y-2$

$y = e^x + 2$

ii) D: all real x

R: $y > 2$

c) $y = 4\sin^{-1}\frac{x}{2}$



d) $y = \frac{1}{\sqrt{9+x^2}}$.

$V = \pi\int_{-3}^{\sqrt{3}} y^2\,dx$

$= \pi\int_{-3}^{\sqrt{3}} \frac{1}{9+x^2}\,dx$

$= \pi\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_{-3}^{\sqrt{3}}$

$= \frac{\pi}{3}\left[\tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1}(-1)\right]$

$= \frac{\pi}{3}\left[\frac{\pi}{6} + \frac{\pi}{4}\right]$

$= \frac{5\pi^2}{36}\ u^3$.

Question 4

a) i) $1 + 3 + 5 + \dots + 2n - 1$

$S_n = \frac{n}{2}(2 + (n-1)2)$

$= \frac{n}{2}(2n)$

$= n^2.$

ii)

Prove $1 + 3 + 5 + \dots 2n - 1 = n^2$

Step 1. Prove true for $n = 1$

LHS = 1

RHS = 1

∴ true for $n = 1$

Step 2. Assume true for $n = k$

ie $1 + 3 + 5 + \dots + 2k - 1 = k^2$

Prove true for $n = k+1$

ie $1 + 3 + 5 + \dots + 2(k+1) - 1 = (k+1)^2$

LHS $= 1 + 3 + \dots + 2k - 1 + 2(k+1) - 1$

$= k^2 + 2(k+1) - 1$

$= k^2 + 2k + 1$

$= (k+1)^2$

$=$ RHS

∴ true for $n = k$ and $n = k+1$

Since true for $n = 1$ it must be true for $n = 2, 3 \dots$

∴ true for all positive integers n by MI.

b) Prove $5^n \geq 1 + 4n$

Step 1. Prove true for $n = 1$

LHS = 5

RHS = 5

∴ true for $n = 1$.

Step 2: Assume true for $n = k$

ie $5^k \geq 1 + 4k$.

Prove true for $n = k+1$

ie $5^{k+1} \geq 1 + 4(k+1)$

$5^{k+1} = 5(5^k)$

$\geq 5(1 + 4k)$

$\geq 5 + 20k.$

∴ $5^{k+1} \geq 5 + 4k$

∴ $5^{k+1} \geq 1 + 4(k+1)$

∴ true for $n = k$ and $n = k+1$

Since true for $n = 1$ it must be true for $n = 2, 3 \dots$

∴ true for all positive integers n by MI

c) Prove $2n + n^3$ is a multiple of 3.

Step 1. Prove true for $n = 1$

$2 + 1 = 3$

$\therefore$ true for $n = 1$.

Step 2: Assume true for $n = k$

ie $2k + k^3 = 3M$ (M is an integer)

Prove true for $n = k+1$

$$2(k+1) + (k+1)^3 = 2k + 2 + k^3 + 3k^2 + 3k + 1$$
$$= k^3 + 2k + 3k^2 + 3k + 3$$
$$= 3M + 3k^2 + 3k + 3$$
$$= 3[M + k^2 + k + 1]$$

$\therefore$ a multiple of 3.

$\therefore$ true for $n = k+1$

Since true for $n = 1$ it must be true for $n = 2, 3, \ldots$

$\therefore$ true for all positive integers n by MI.

Question 5

a) i) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

ii) $\sin(105) = \sin 45\cos 60 + \cos 45 \sin 60$

$= \frac{1}{\sqrt{2}}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{2}+\sqrt{6}}{4}$

b) $\sin 2\beta = 2\sin\beta\cos\beta$

$= 2\frac{\sin\beta}{\cos\beta}.\cos^2\beta$

$= 2\tan\beta . \frac{1}{\sec^2\beta}$

$= \frac{2\tan\beta}{\sec^2\beta}$

$= \frac{2\tan\beta}{1+\tan^2\beta}$.

c) $\tan(\frac{\pi}{4}+\alpha) = \frac{\tan\frac{\pi}{4}+\tan\alpha}{1-\tan\frac{\pi}{4}\tan\alpha}$

$= \frac{1+\frac{\sin\alpha}{\cos\alpha}}{1-\frac{\sin\alpha}{\cos\alpha}}$

$= \frac{\frac{\cos\alpha+\sin\alpha}{\cos\alpha}}{\frac{\cos\alpha-\sin\alpha}{\cos\alpha}}$

$= \frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$.

d) $\sin\beta + \cos\beta = 1$

$\tan\frac{\theta}{2} = t$

$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1$

$2t + 1 - t^2 = 1+t^2$

$2t^2 - 2t = 0$

$t(t-1) = 0$

$t = 0$ $\qquad$ $t = 1$

$\tan\frac{\theta}{2} = 0$ $\qquad$ $\tan\frac{\theta}{2} = 1$

$\frac{\theta}{2} = 0, \pi$ $\qquad$ $\frac{\theta}{2} = \frac{\pi}{4}, \frac{5\pi}{4}$

$\theta = 0, 2\pi$ $\qquad$ $\theta = \frac{\pi}{2}, \frac{5\pi}{2}$

$\therefore$ Solutions

$\theta = 0, \frac{\pi}{2}, 2\pi$

Question 6

a) $x = 6t \quad y = 2t^2$

$\frac{x}{6} = t$

$y = 2\left(\frac{x}{6}\right)^2$

$y = \frac{x^2}{18}$

b) $x^2 = 8y \quad (4t, 2t^2)$

$y = \frac{1}{8}x^2 \quad M_T = \frac{1}{4}(4t)$

$y' = \frac{1}{4}x \quad M_N = \frac{=}{} t \; -\frac{1}{t}$

$y - 2t^2 = \frac{-1}{t}(x - 4t)$

$ty - 2t^3 = -x + 4t$

$x + ty = 2t^3 + 4t$

c) i) $y = x^2 + 1 \quad T(t, t^2+1)$

$y' = 2x$

$M_T = 2t$

$y - (t^2+1) = 2t(x - t)$

$y - t^2 - 1 = 2tx - 2t^2$

$y = 2tx - t^2 + 1$

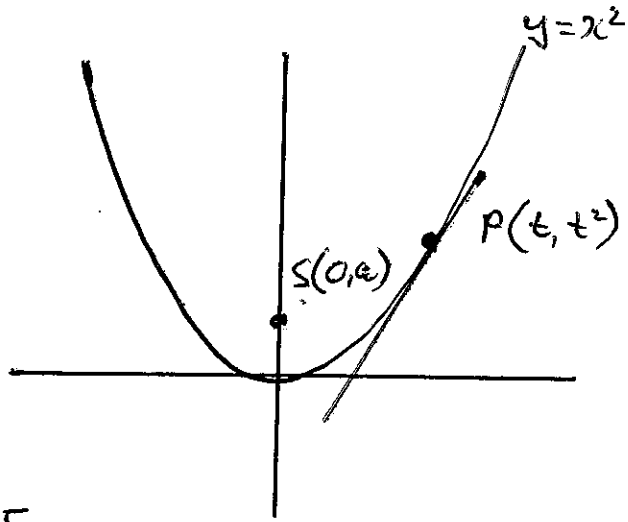
ii) $y = 1$ TANGENT AT VERTEX

$1 = 2tx - t^2 + 1$

$2tx = t^2$

$x = t/2 \quad R(t/2, 1)$

d)



i) Focus

$x^2 = y$

$x^2 = 4(\frac{1}{4})y$

$\therefore a = \frac{1}{4}$

Focus $(0, \frac{1}{4})$

ii) $y = x^2$

$y' = 2x$

M_T at $P = 2t$

$M_\perp$ through $S = -\frac{1}{2t}$

$y - \frac{1}{4} = -\frac{1}{2t}(x)$

$2ty - \frac{t}{2} = -x$

$4ty + 2x = t$

iii) $y - 2tx + t^2 = 0 \quad —— ①$

$4ty + 2x = t \quad —— ②$

From ①

$y = 2tx - t^2 \quad —— ③$

sub into ②

$4t(2tx - t^2) + 2x = t$

$8t^2x - 4t^3 + 2x = t$

$x = \frac{4t^3 + t}{8t^2 + 2}$

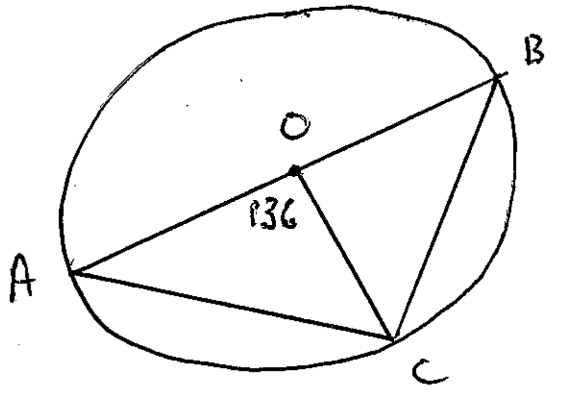
$= \frac{t}{2}$

when $x = \frac{t}{2} \quad y = 2t(\frac{t}{2}) - t^2$

LOCUS $y = 0$

Question 7

a)

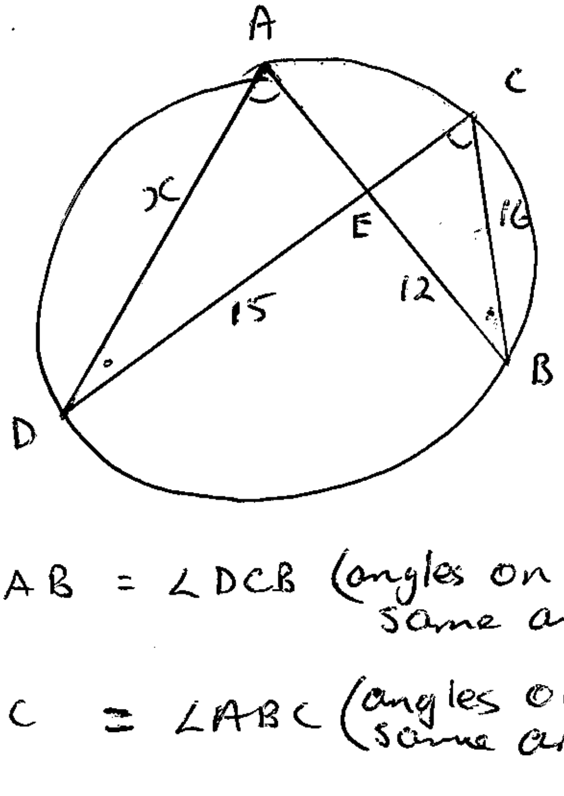


$\angle OCB + \angle OBC = 136°$ (Ext $\angle$ of Δ)

$\angle OCB = \angle OBC$ (ΔOBC is isos)

$\therefore \angle OCB = 68°$.

b)



$\angle DAB = \angle DCB$ (angles on same arc)

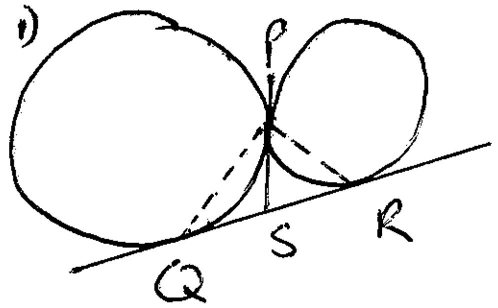
$\angle ADC = \angle ABC$ (angles on same arc)

$\therefore \Delta AED \,|||\, \Delta CEB$ (equiangular)

$\therefore \frac{x}{16} = \frac{15}{12}$ (Corresponding sides in similar triangles)

$x = 20$

c) i)



ii) $PS = SQ$ (Tangents from an external pt)

$PS = SR$ (")

$\therefore SQ = SR$

$\therefore S$ is midpoint of QR

iii) Let $\angle SRP = x$

$\therefore \angle SPR = x$ (ΔSPR is isos)

Let $\angle SQP = y$

$\therefore \angle SPQ = y$ (ΔSQP is isos)

$x + x + y + y = 180°$ (angle sum of Δ)

$\therefore x + y = 90$

$\therefore \angle QPR = 90°$.